Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

January 2017: Problem 4 Solution

Exercise. Let $a, b \in \mathbb{R}$, $a < b$.

(a) Define what it means for a function $f : [a, b] \to \mathbb{C}$ to be "absolutely continuous"

Solution.

f is **absolutely continuous on** $[a, b]$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ s.t. $(a_j, b_j) \subseteq [a, b]$ for all j,

$$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^n |f(b_j) - f(a_j)| < \epsilon$$

(b) Prove, using the definition of absolute continuity, that the product of two absolutely continuous functions is absolutely continuous.

Solution.

Let f and g be absolutely continuous functions.
Since f, g continuous, they are bounded on $[a, b]$

$\implies \exists M, N' \in \mathbb{N}$ s.t. $|f(x)| \leq M$ and $|g(x)| \leq N$ for all $x \in [a, b]$

Since f, g absolutely continuous, for any $\epsilon > 0$,

$$\exists \delta_1 > 0 \text{ s.t.} \quad \sum_1^n (b_j - a_j) < \delta_1 \implies \sum_1^n (f(b_j) - f(a_j)) < \frac{\epsilon}{2N},$$

$$\text{and } \exists \delta_2 > 0 \text{ s.t.} \quad \sum_1^n (b_j - a_j) < \delta_2 \implies \sum_1^n (g(b_j) - g(a_j)) < \frac{\epsilon}{2M}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $\sum_1^n (b_j - a_j) < \delta$,

$$\begin{aligned}
\sum_1^n \left| f(b_j)g(b_j) - f(a_j)g(a_j) \right| &= \sum_1^n \left| f(b_j)g(b_j) - f(b_j)g(a_j) + f(b_j)g(a_j) - f(a_j)g(a_j) \right| \\
&\leq \sum_1^n \left[\left| f(b_j) \right| \cdot \left| g(b_j) - g(a_j) \right| + \left| g(a_j) \right| \cdot \left| f(b_j) - f(a_j) \right| \right] \\
&\leq \sum_1^n \left[M \left| g(b_j) - g(a_j) \right| + N \left| f(b_j) - f(a_j) \right| \right] \\
&= M \sum_1^n \left| g(b_j) - g(a_j) \right| + N \sum_1^n \left| f(b_j) - f(a_j) \right| \\
&< M \left(\frac{\epsilon}{2M} \right) + N \left(\frac{\epsilon}{2N} \right) \\
&= \epsilon
\end{aligned}$$

Thus, $f(x)g(x)$ is absolutely continuous.